



Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®

J. Math. Anal. Appl. 288 (2003) 20–27

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

On the forced oscillation of systems of neutral parabolic differential equations with deviating arguments[☆]

Wei Nian Li^{a,b,c,*} and Fanwei Meng^{a,b}

^a Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, People's Republic of China

^b Department of Mathematics, Qufu Normal University, Shandong 273165, People's Republic of China

^c Department of Mathematics, Binzhou Normal College, Shandong 256604, People's Republic of China

Received 16 April 2002

Submitted by K. Lurie

Abstract

Sufficient conditions are established for the forced oscillation of systems of neutral parabolic differential equations with deviating arguments. The main results are illustrated by some examples.
© 2003 Elsevier Inc. All rights reserved.

Keywords: Forced oscillation; System; Parabolic differential equation; Deviating argument; Neutral type

1. Introduction

In the past decade, the fundamental theory of partial functional differential equations has been investigated extensively. We refer the reader to the monograph by Wu [1]. Recently, a few results on the oscillation theory for systems of partial functional differential equations were established in [2–9]. However, using the approach in these papers, it is impossible to obtain the forced oscillation of systems of partial functional differential equations.

Our aim in this paper is to study the forced oscillation of systems of neutral parabolic differential equations with deviating arguments of the form

[☆] This work is supported by the Natural Science Foundation of Shandong Province, China (Y2001A03).

* Corresponding author.

E-mail address: wnli@xinhuanet.com (W.N. Li).

$$\begin{aligned}
\frac{\partial}{\partial t} \left(u_i(x, t) + \sum_{r=1}^s \lambda_{ir}(t) u_i(x, \rho_{ir}(t)) \right) &= \sum_{k=1}^m a_{ik}(t) \Delta u_k(x, t) \\
&+ \sum_{k=1}^m b_{ik}(t) \Delta u_k(x, \tau_{ik}(t)) - c_i \left(x, t, (u_k(x, t))_{k=1}^m, (u_k(x, \sigma_k(t)))_{k=1}^m \right) \\
&+ f_i(x, t), \quad (x, t) \in \Omega \times [0, \infty) \equiv G, \quad i = 1, 2, \dots, m,
\end{aligned} \tag{1}$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and

$$\Delta u_i(x, t) = \sum_{r=1}^n \frac{\partial^2 u_i(x, t)}{\partial x_r^2}, \quad i = 1, 2, \dots, m.$$

We assume throughout this paper that

- (A1) $\lambda_{ir} \in C^1([0, \infty); [0, \infty))$, $a_{ik}, b_{ik} \in C([0, \infty); R)$, $a_{ii}(t) > 0$, and $b_{ii}(t) > 0$, $i = 1, 2, \dots, m$, $k = 1, 2, \dots, m$, $r = 1, 2, \dots, s$;
(A2) $\rho_{ir}, \tau_{ik}, \sigma_k \in C([0, \infty); R)$, $\rho_{ir}(t) \leq t$, $\tau_{ik}(t) \leq t$, $\sigma_k(t) \leq t$, and $\lim_{t \rightarrow \infty} \rho_{ir}(t) = \lim_{t \rightarrow \infty} \tau_{ik}(t) = \lim_{t \rightarrow \infty} \sigma_k(t) = \infty$, $i = 1, 2, \dots, m$, $r = 1, 2, \dots, s$, $k = 1, 2, \dots, m$;
(A3) $c_i \in C(\bar{G} \times R^{2m}; R)$, and

$$c_i(x, t, \xi_1, \dots, \xi_i, \dots, \xi_m, \eta_1, \dots, \eta_i, \dots, \eta_m) \begin{cases} \geq 0, & \text{if } \xi_i \text{ and } \eta_i \in (0, \infty), \\ \leq 0, & \text{if } \xi_i \text{ and } \eta_i \in (-\infty, 0), \end{cases}$$

$i = 1, 2, \dots, m$;

- (A4) $f_i \in C(\bar{G}; R)$, $i = 1, 2, \dots, m$.

Consider the following boundary condition:

$$\frac{\partial u_i(x, t)}{\partial N} = \psi_i(x, t), \quad (x, t) \in \partial\Omega \times [0, \infty), \quad i = 1, 2, \dots, m, \tag{2}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\psi_i(x, t)$ is a continuous function on $\partial\Omega \times [0, \infty)$, $i = 1, 2, \dots, m$.

Definition 1.1. The vector function $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ is said to be a solution of the problem (1), (2) if it satisfies (1) in $G = \Omega \times [0, \infty)$ and boundary condition (2).

Definition 1.2. The vector solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ of the problem (1), (2) is said to oscillate in the domain $G = \Omega \times [0, \infty)$ if at least one of its nontrivial component oscillates in G . Otherwise, the vector solution $u(x, t)$ is said to be nonoscillatory.

Definition 1.3. The vector solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ of the problem (1), (2) is said to oscillate strongly in the domain $G = \Omega \times [0, \infty)$ if each of its nontrivial component oscillates in G .

2. Main results

For convenience, we introduce the following notations:

$$\begin{aligned} U_i(t) &= \int_{\Omega} u_i(x, t) dx, & \Psi_i(t) &= \int_{\partial\Omega} \psi_i(x, t) dS, & F_i(t) &= \int_{\Omega} f_i(x, t) dx, \\ H_i(t) &= F_i(t) + \sum_{k=1}^m a_{ik}(t) \Psi_k(t) + \sum_{k=1}^m b_{ik}(t) \Psi_k(\tau_{ik}(t)), & t &\geq 0, \quad i = 1, 2, \dots, m, \end{aligned}$$

where dS is the surface element on $\partial\Omega$.

Lemma 2.1. Suppose that $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ is a solution of the problem (1), (2) in G . If there exists some $i_0 \in \{1, 2, \dots, m\}$ such that $u_{i_0}(x, t) > 0$, $t \geq t_0 \geq 0$, then $U_{i_0}(t)$ satisfies the neutral differential inequality

$$\left(V(t) + \sum_{r=1}^s \lambda_{i_0 r}(t) V(\rho_{i_0 r}(t)) \right)' \leq H_{i_0}(t). \quad (3)$$

Proof. From the condition (A2) we easily obtain that there exists a number $t_1 \geq t_0$ such that $u_{i_0}(x, t) > 0$, $u_{i_0}(x, \rho_{i_0 r}(t)) > 0$, $u_{i_0}(x, \tau_{i_0 k}(t)) > 0$ and $u_{i_0}(x, \sigma_{i_0}(t)) > 0$ in $\Omega \times [t_1, \infty)$, $k = 1, 2, \dots, m$, $r = 1, 2, \dots, s$. Consider the following equation:

$$\begin{aligned} \frac{\partial}{\partial t} \left(u_{i_0}(x, t) + \sum_{r=1}^s \lambda_{i_0 r}(t) u_{i_0}(x, \rho_{i_0 r}(t)) \right) &= \sum_{k=1}^m a_{i_0 k}(t) \Delta u_k(x, t) \\ &+ \sum_{k=1}^m b_{i_0 k}(t) \Delta u_k(x, \tau_{i_0 k}(t)) - c_{i_0} \left(x, t, (u_k(x, t))_{k=1}^m, (u_k(x, \sigma_k(t)))_{k=1}^m \right) \\ &+ f_{i_0}(x, t), \quad (x, t) \in \Omega \times [0, \infty) \equiv G. \end{aligned} \quad (4)$$

Integrating (4) with respect to x over the domain Ω , we have

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} u_{i_0}(x, t) dx + \sum_{r=1}^s \lambda_{i_0 r}(t) \int_{\Omega} u_{i_0}(x, \rho_{i_0 r}(t)) dx \right) \\ = \sum_{k=1}^m a_{i_0 k}(t) \int_{\Omega} \Delta u_k(x, t) dx + \sum_{k=1}^m b_{i_0 k}(t) \int_{\Omega} \Delta u_k(x, \tau_{i_0 k}(t)) dx \\ - \int_{\Omega} c_{i_0} \left(x, t, (u_k(x, t))_{k=1}^m, (u_k(x, \sigma_k(t)))_{k=1}^m \right) dx + \int_{\Omega} f_{i_0}(x, t) dx, \quad t \geq t_1. \end{aligned} \quad (5)$$

Green's formula and (2) yield

$$\int_{\Omega} \Delta u_k(x, t) dx = \int_{\partial\Omega} \frac{\partial u_k(x, t)}{\partial N} dS = \int_{\partial\Omega} \psi_k(x, t) dS = \Psi_k(t) \quad (6)$$

and

$$\begin{aligned} \int_{\Omega} \Delta u_k(x, \tau_{i_0 k}(t)) dx &= \int_{\partial \Omega} \frac{\partial u_k(x, \tau_{i_0 k}(t))}{\partial N} dS \\ &= \int_{\partial \Omega} \psi_k(x, \tau_{i_0 k}(t)) dS = \Psi_k(\tau_{i_0 k}(t)), \quad t \geq t_1, \quad k = 1, 2, \dots, m. \end{aligned} \quad (7)$$

From the condition (A3) we have $c_{i_0}(x, t, (u_k(x, t))_{k=1}^m, (u_k(x, \sigma_k(t)))_{k=1}^m) > 0$; then combining (5)–(7), we have

$$\begin{aligned} &\left(U_{i_0}(t) + \sum_{r=1}^s \lambda_{i_0 r}(t) U_{i_0}(\rho_{i_0 r}(t)) \right)' \\ &\leq F_{i_0}(t) + \sum_{k=1}^m a_{i_0 k}(t) \Psi_k(t) + \sum_{k=1}^m b_{i_0 k}(t) \Psi_k(\tau_{i_0 k}(t)), \quad t \geq t_1, \end{aligned}$$

which shows that $U_{i_0}(t) > 0$ is a positive solution of the inequality (3). The proof is complete. \square

The proof of the following lemma is similar to that of Lemma 2.1 and we omit it.

Lemma 2.2. Suppose that $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ is a solution of the problem (1), (2) in G . If there exists some $i_0 \in \{1, 2, \dots, m\}$ such that $u_{i_0}(x, t) < 0$, $t \geq t_0 \geq 0$, then $U_{i_0}(t)$ satisfies the neutral differential inequality

$$\left(V(t) + \sum_{r=1}^s \lambda_{i_0 r}(t) V(\rho_{i_0 r}(t)) \right)' \geq H_{i_0}(t). \quad (8)$$

Theorem 2.1. If there exists some $i_0 \in \{1, 2, \dots, m\}$ such that inequality (3) has no eventually positive solutions and inequality (8) has no eventually negative solutions, then every solution of the problem (1), (2) is oscillatory in G .

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ of the problem (1), (2). It is obvious that $|u_i(x, t)| > 0$ for $t \geq t_0 \geq 0$, $i = 1, 2, \dots, m$, then $u_{i_0}(x, t) > 0$ or $u_{i_0}(x, t) < 0$, $t \geq t_0$.

If $u_{i_0}(x, t) > 0$, $t \geq t_0$, using Lemma 2.1 we obtain that $U_{i_0}(t) > 0$ is a solution of inequality (3), which is a contradiction.

If $u_{i_0}(x, t) < 0$, $t \geq t_0$, using Lemma 2.2 we obtain that $U_{i_0}(t) < 0$ is a solution of inequality (8), which is a contradiction. This completes the proof. \square

Lemma 2.3. If

$$\liminf_{t \rightarrow \infty} \int_{t_1}^t H_{i_0}(s) ds = -\infty, \quad t_1 \geq t_0, \quad (9)$$

then inequality (3) has no eventually positive solutions.

Proof. Assume to a contrary that (3) has a positive solution $V(t)$; then there exists $t_0 \geq 0$ such that $V(t) > 0$, $V(\rho_{i_0 r}(t)) > 0$, $t \geq t_0$, $r = 1, 2, \dots, s$. Integrating (3) over the interval $[t_1, t]$, $t_1 \geq t_0$, we have

$$V(t) + \sum_{r=1}^s \lambda_{i_0 r}(t) V(\rho_{i_0 r}(t)) \leq C + \int_{t_1}^t H_{i_0}(s) ds, \quad (10)$$

where C is a constant.

Taking $t \rightarrow \infty$, from (10) we have

$$\liminf_{t \rightarrow \infty} \left[V(t) + \sum_{r=1}^s \lambda_{i_0 r}(t) V(\rho_{i_0 r}(t)) \right] = -\infty,$$

which contradicts with the assumption that $V(t) > 0$. The proof is complete. \square

Similarly we have

Lemma 2.4. *If*

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t H_{i_0}(s) ds = \infty, \quad t_1 \geq t_0, \quad (11)$$

then inequality (8) has no eventually negative solutions.

Using Theorem 2.1, Lemma 2.3 and Lemma 2.4, we immediately obtain the following theorem.

Theorem 2.2. *If there exists some $i_0 \in \{1, 2, \dots, m\}$ such that (9) and (11) hold, then every solution of the problem (1), (2) is oscillatory in G .*

Using the above oscillation results, it is not difficult to derive the following strong oscillation conclusions.

Theorem 2.3. *Suppose that for all $i \in \{1, 2, \dots, m\}$*

$$\left(V(t) + \sum_{r=1}^s \lambda_{ir}(t) V(\rho_{ir}(t)) \right)' \leq H_i(t) \quad (12)$$

has no eventually positive solutions and

$$\left(V(t) + \sum_{r=1}^s \lambda_{ir}(t) V(\rho_{ir}(t)) \right)' \geq H_i(t) \quad (13)$$

has no eventually negative solutions. Then every solution of the problem (1), (2) oscillates strongly in G .

Theorem 2.4. Suppose that for all $i \in \{1, 2, \dots, m\}$

$$\liminf_{t \rightarrow \infty} \int_{t_1}^t H_i(s) ds = -\infty, \quad t_1 \geq t_0, \quad (14)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t H_i(s) ds = \infty, \quad t_1 \geq t_0 \quad (15)$$

hold. Then every solution of the problem (1), (2) oscillates strongly in G .

3. Examples

In this section, we give two illustrative examples.

Example 3.1. Consider the system of neutral parabolic differential equations

$$\begin{cases} \frac{\partial}{\partial t} [u_1(x, t) + u_1(x, t - \pi)] = \Delta u_1(x, t) + e^\pi \Delta u_1(x, t - \pi) + \Delta u_2(x, t) \\ \quad + e^{\pi/2} \Delta u_2(x, t - 3\pi/2) - u_1(x, t) - u_1(x, t - \pi) + e^{-\pi} e^t \sin t \sin x \\ \quad + 2e^t \cos t \sin x, \\ \frac{\partial}{\partial t} [u_2(x, t) + e^{\pi/2} u_2(x, t - \pi/2)] = e^{-\pi} \Delta u_1(x, t) + \Delta u_1(x, t - \pi) \\ \quad + \Delta u_2(x, t) + e^\pi \Delta u_2(x, t - 3\pi/2) - u_2(x, t) - u_2(x, t - \pi/2) \\ \quad + 4e^t \sin t \sin x, \quad (x, t) \in (0, \pi) \times [0, \infty), \end{cases} \quad (16)$$

with boundary condition

$$\begin{cases} -\frac{\partial u_1(0, t)}{\partial x} = \frac{\partial u_1(\pi, t)}{\partial x} = -e^t \cos t, \\ -\frac{\partial u_2(0, t)}{\partial x} = \frac{\partial u_2(\pi, t)}{\partial x} = -e^t \sin t, \quad t \geq 0. \end{cases} \quad (17)$$

Here $\Omega = (0, \pi)$, $n = 1$, $m = 2$, $s = 1$, $\lambda_{11}(t) = 1$, $\lambda_{21}(t) = e^{\pi/2}$, $\rho_{11}(t) = t - \pi$, $\rho_{21}(t) = t - \pi/2$, $a_{11}(t) = 1$, $a_{12}(t) = 1$, $a_{21}(t) = e^{-\pi}$, $a_{22}(t) = 1$, $b_{11}(t) = e^\pi$, $b_{12}(t) = e^{\pi/2}$, $b_{21}(t) = 1$, $b_{22}(t) = e^\pi$, $\tau_{11}(t) = \tau_{21}(t) = t - \pi$, $\tau_{12}(t) = \tau_{22}(t) = t - 3\pi/2$, $c_i(x, t, u_1(x, t), u_2(x, t), u_1(x, \sigma_1(t)), u_2(x, \sigma_2(t))) = u_i(x, t) + u_i(x, \sigma_i(t))$, $i = 1, 2$; $\sigma_1(t) = t - \pi$, $\sigma_2(t) = t - \pi/2$, $\psi_1(x, t) = -e^t \cos t$, $\psi_2(x, t) = -e^t \sin t$, $f_1(x, t) = e^{-\pi} e^t \sin t \sin x + 2e^t \cos t \sin x$, $f_2(x, t) = 4e^t \sin t \sin x$.

It is obvious that $\Psi_1(t) = -2e^t \cos t$, $\Psi_2(t) = -2e^t \sin t$, $\Psi_1(\tau_{11}(t)) = \Psi_1(\tau_{21}(t)) = 2e^{-\pi} e^t \cos t$, $\Psi_2(\tau_{12}(t)) = \Psi_2(\tau_{22}(t)) = -2e^{-3\pi/2} e^t \cos t$; then

$$F_1(t) = \int_{\Omega} f_1(x, t) dx = \int_0^\pi f_1(x, t) dx = 2e^t (e^{-\pi} \sin t + 2 \cos t),$$

$$F_2(t) = \int_{\Omega} f_2(x, t) dx = \int_0^\pi f_2(x, t) dx = 8e^t \sin t,$$

$$\begin{aligned}
H_1(t) &= F_1(t) + a_{11}(t)\Psi_1(t) + a_{12}(t)\Psi_2(t) + b_{11}(t)\Psi_1(\tau_{11}(t)) + b_{12}(t)\Psi_2(\tau_{12}(t)) \\
&= 2e^t[(e^{-\pi} - 1)\sin t + (2 - e^{-\pi})\cos t], \\
H_2(t) &= F_2(t) + a_{21}(t)\Psi_1(t) + a_{22}(t)\Psi_2(t) + b_{21}(t)\Psi_1(\tau_{21}(t)) + b_{22}(t)\Psi_2(\tau_{22}(t)) \\
&= 2e^t(3\sin t - e^{-\pi/2}\cos t).
\end{aligned}$$

Hence

$$\liminf_{t \rightarrow \infty} \int_{t_1}^t H_1(s) ds = -\infty, \quad \limsup_{t \rightarrow \infty} \int_{t_1}^t H_1(s) ds = \infty$$

and

$$\liminf_{t \rightarrow \infty} \int_{t_1}^t H_2(s) ds = -\infty, \quad \limsup_{t \rightarrow \infty} \int_{t_1}^t H_2(s) ds = \infty,$$

which shows that all the conditions of Theorem 2.4 are fulfilled. Then every solution of the problem (16), (17) oscillates strongly in $(0, \pi) \times [0, \infty)$. In fact, $u_1(x, t) = e^t \cos t \sin x$, $u_2(x, t) = e^t \sin t \sin x$ is such a solution.

Example 3.2. Consider the system of neutral parabolic differential equations

$$\begin{cases}
\frac{\partial}{\partial t}[u_1(x, t) + e^\pi u_1(x, t - \pi)] = \Delta u_1(x, t) + e^\pi \Delta u_1(x, t - \pi) \\
\quad + \Delta u_2(x, t) + (-e^{3\pi/2}) \Delta u_2(x, t - 3\pi/2) - u_1(x, t) - u_1(x, t - \pi/2) \\
\quad + (\cos t + e^{-\pi/2} \sin t) e^t \sin x, \\
\frac{\partial}{\partial t}[u_2(x, t) + e^{\pi/2} u_2(x, t - \pi/2)] = \Delta u_1(x, t) + e^\pi \Delta u_1(x, t - \pi) \\
\quad + \Delta u_2(x, t) + e^{2\pi/3} \Delta u_2(x, t - \pi) - u_2(x, t) - u_2(x, t - \pi/3) \\
\quad + 2(2 + e^{-\pi/3}) e^t \cos x, \quad (x, t) \in (0, \pi) \times [0, \infty),
\end{cases} \quad (18)$$

with boundary condition

$$\begin{cases}
-\frac{\partial u_1(0, t)}{\partial x} = \frac{\partial u_1(\pi, t)}{\partial x} = -e^t \cos t, \\
-\frac{\partial u_2(0, t)}{\partial x} = \frac{\partial u_2(\pi, t)}{\partial x} = 0, \quad t \geq 0.
\end{cases} \quad (19)$$

It is easy to see that $H_1(t) = 2e^t(\cos t + e^{-\pi/2} \sin t)$, $H_2(t) = 0$. Therefore,

$$\liminf_{t \rightarrow \infty} \int_{t_1}^t H_1(s) ds = -\infty, \quad \limsup_{t \rightarrow \infty} \int_{t_1}^t H_1(s) ds = \infty.$$

Then, using Theorem 2.2, we obtain that every solution of the problem (18), (19) oscillates in $(0, \pi) \times [0, \infty)$. In fact, $u_1(x, t) = e^t \cos t \sin x$, $u_2(x, t) = e^t \cos x$ is such a solution.

References

- [1] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

- [2] Y.K. Li, Oscillation of systems of hyperbolic differential equations with deviating arguments, *Acta Math. Sinica* 40 (1997) 100–105, in Chinese.
- [3] W.N. Li, B.T. Cui, Oscillations of systems of neutral delay parabolic equations, *Demonstratio Math.* 31 (1998) 813–824.
- [4] W.N. Li, B.T. Cui, Oscillation for systems of parabolic equations of neutral type, *Southeast Asian Bull. Math.* 23 (1999) 447–456.
- [5] W.N. Li, Oscillation for systems of partial differential equations with functional arguments, *Demonstratio Math.* 32 (1999) 521–530.
- [6] W.N. Li, L. Debnath, Oscillation of a systems of delay hyperbolic differential equations, *Int. J. Appl. Math.* 2 (2000) 417–431.
- [7] W.N. Li, B.T. Cui, Oscillation for systems of neutral delay hyperbolic differential equations, *Indian J. Pure Appl. Math.* 31 (2000) 933–948.
- [8] W.N. Li, Oscillation properties for systems of hyperbolic differential equations of neutral type, *J. Math. Anal. Appl.* 248 (2000) 369–384.
- [9] W.N. Li, F.W. Meng, Oscillation for systems of neutral partial differential equations with continuous distributed deviating arguments, *Demonstratio Math.* 34 (2001) 619–633.